Available online at www.sciencedirect.com

SCIENCE DIRECT

ELSEVIER

Journal of APPLIED MATHEMATICS AND MECHANICS

www.elsevier.com/locate/jappmathmech

Journal of Applied Mathematics and Mechanics 69 (2005) 173–182

MODELLING OF THE CONTROLLED MOTION OF A POINT ON A PLANE†

S. V. VOLKOV

Moscow

e-mail: svvolkov@sci.pfu.edu.ru

(*Received* 8 *October* 2003)

A solution of the problem of feedback control of the motion of a point on a plane is presented. The equations of the control programme (the objective) are set up as a system of differential equations with a given set of singular trajectories in the domain of admissible positions of the controlled point, as well as a given topological structure of the partition into trajectories. These equations define the vector field of velocities of the programmed motions of the point and are used to find the corresponding control forces.

1. INTRODUCTION

The main operation when solving problems in the mechanics of a controllable body is forming the objective of the control, which presupposes setting up a system of equations (both finite and differential) of the mechanical constraints whose realization will guarantee that the motion of the body will have the required properties. These equations are used to obtain the control forces by solving the corresponding inverse problem of dynamics [1]. As the techniques for solving inverse problems of dynamic have developed, the number and diversity of the required properties of the motion has increased. In classical problems for a point mass, these concerned the existence of a given trajectory of the point, or the fact that the trajectory belonged to some family of curves (the problems of Newton, Kepler, Bertrand, Meshcherskii, and Zhukovskii [2, 3]), and for a mechanical system, the existence of given particular integrals of the equations of motion (the problems of Ermakov, Suslov, Goryachev, and Chaplygin [2, 3]). The properties of the motions are described more fully by differential equations. They may be used to formalize information about both individual trajectories and manifolds of such trajectories, as well as the nature of their stability, attraction domains, and other properties. Methods for formulating systems of differential equations that have a given family of singular integral manifolds and given properties of partition of the space of variables into trajectories [4–7]‡ provide the mathematical basis for the modern approach to solving inverse problems of dynamics in the most general settings [8–10]. In this paper, the control of the motion of a point mass will be synthesized using a method described in [11] for constructing systems of equations of the form

$$\dot{x} = X(x, y), \quad \dot{y} = Y(x, y) \tag{1.1}$$

of class $C_1(\Omega)$, which have given trajectories Γ_i: $\omega_i(x, y) = 0$ $(i = 1, \dots, n)$, among which there may be non-closed curves, limit cycles, simple equilibrium states (foci, nodes or saddle points) and compound equilibrium states, and for which the partition of the domain Ω of the (x, y) plane into trajectories has a given structure. The essence of the method is to reduce the solution of the problem to constructing, first, two orthogonal vector fields of directions of comparison corresponding to a set of given trajectories of the desired system of equations (1.1), and, second, two functions which are scalar products of the vector of the right-hand sides of the desired system of equations (1.1) and the vectors of these fields

†*Prikl. Mat. Mekh.* Vol. 69, No. 2, pp. 191–201, 2005.

0021–8928/$—see front matter.

doi: 10.1016/j.jappmathmech.2005.03.002

‡See also VOLKOV, S. V., Construction of dynamical systems given the properties of the behavior of the solutions. Candidate dissertation 01.02.01. Moscow, 1981.

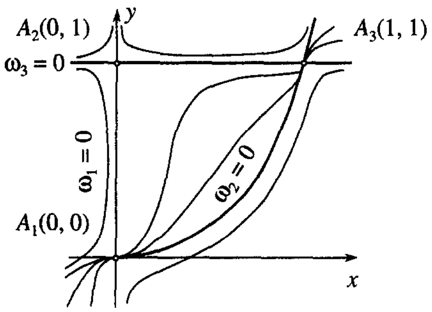

Fig. 1

of comparison directions, and which correspond to the given topological structure of the partition of Ω into trajectories. The theory underlying these constructions consists of the fundamental assumptions of the qualitative theory of systems of equations of type (1.1) of class C_1 [1, 2] and modifications of the methods of Yerugin [14] and Frommer for investigating the singular points of an ordinary differential equation [15].

2. FORMULATION OF THE PROBLEM

It is required to find a force under whose action the motion of a point M of mass m in a domain Ω has the following properties.

Property A. The point moves from an initial position $A_1(0, 0)$ (Fig. 1) to a final position $A_3(1, 1)$ without leaving the domain Ω bounded by the curves

$$\omega_1 \equiv x = 0, \quad \omega_2 \equiv y - x^3 = 0, \quad \omega_3 \equiv y - 1 = 0 \tag{2.1}$$

Property B. The point has zero velocity at its initial and final points, where its trajectories touch the curve $\omega_2 = 0$ (Fig. 1).

A prototype of such a point would be, in particular, the centre of mass of the grip of a robot manipulator or any of its modules.

3. SOLUTION OF THE PROBLEM

We will begin by formulating a programme for the motion of the point M as a system of kinematic equations (1.1), to which the topological structure of the partition of Ω into trajectories, shown graphically in Fig. 1, corresponds. To do this, the steps described below are taken.

3.1. *Construction of the vectors of comparison directions.* The vectors $\mathbf{n}$ and τ of the auxiliary fields of comparison directions are used as the vectors of the local basis in which the direction of the vector of right-hand sides of the desired system (1.1) is given. These vectors, corresponding to a given set of curves $\{\Gamma_i\colon \omega_i = 0 \mid i = 1, 2, 3\}$, are given the following properties

$$(\mathbf{n} \cdot \operatorname{grad} \omega_i)|_{\Gamma_i} = 0, i = 1, 2, 3, \mathbf{n} \neq 0 \text{ and } n_y \geq 0 \text{ in the domain } \Omega \backslash (A_1 \cup A_2 \cup A_3)$$

$$\tau_x = n_y, \tau_y = -n_x,$$

where A_1, A_2 and A_3 are equilibrium points of system (1.1).

To construct such vectors $\mathbf{n}$ and $\boldsymbol{\tau}$, we use the vectors [11]

$$\mathbf{n}_1 = (1, 0), \quad \mathbf{n}_2 = (-3x^2, 1), \quad \mathbf{n}_3 = (0, 1) \tag{3.1}$$

corresponding to curves (2.1), and the vectors

$$\mathbf{m}_1 = (-xy, x^2), \quad \mathbf{m}_2 = (-x(y-1), x^2), \quad \mathbf{m}_3 = (-(x-1)(y-1), (x-1)^2) \tag{3.2}$$

corresponding to the equilibrium states A_1, A_2 and A_3, Throughout, the subscripts i, j and k take values 1, 2, 3, unless otherwise states.

We define

$$\mathbf{n} = \sum_i \lambda_i \mathbf{n}_i \prod_{j, j \neq i} \omega_j^{\alpha_{ij}} + \sum_k \mu_k \mathbf{m}_k \prod_i \omega_i^{\beta_{ki}} \tag{3.3}$$

where the arbitrary non-negative coefficients $\lambda_i(x, y)$ and $\mu_k(x, y)$ and arbitrary natural numbers α_{ij} and β_{ki} are chosen so that the following conditions are satisfied.

1. In a fairly small neighbourhood of the point A_k the term $\mu_k \mathbf{m}_k \prod_i \omega_i^{\beta_{ki}}$ is the principal term of (3.3) (ensuring the existence of only the required exceptional directions of the equilibrium state A_k of the vector field $\mathbf{n}$), while near the curve Γ_i the principal term is the corresponding term $\lambda_i \mathbf{n}_i \prod_{j, j \neq i} \omega_j^{\alpha_{ij}}$ (making it possible to guarantee the desired type of separatrix Γ_i).

2. In a fairly small neighbourhood of the curve Γ_i the term $\lambda_i \mathbf{n}_i \prod_{j, j \neq i} \omega_j^{\alpha_{ij}}$ corresponding to that curve is the principal term of (3.3).

Using the notation $\deg_A f(x, y)$ for the least degree of the terms in the expansion of f in a series of powers of $x - a$ and $y - b$ at the point $A(a, b)$, we define numbers $\psi_{ik} = \deg_{A_k}(\partial\omega_i/\partial y)$ for $i \in I_{A_k} = \{i: \omega_i(A_k) = 0\}$ and note that, if λ_i, μ_k, α_{ij} and β_{ki} satisfy the inequalities

$$\deg_{A_k}(\lambda_i) + \sum_j \alpha_{ij} \deg_{A_k}(\omega_j) + \psi_{ik} > \beta_k + \deg_{A_k}(\mu_k) + 2$$

$$\deg_{A_k}(\mu_m) + \sum_j \beta_{mj} \deg_{A_k}(\omega_j) > \beta_k + 2$$

the conditions 1 and 2 will be satisfied.

3.2. *Analytical definition of the structure of the phase portrait.* To express properties A and B in mathematical form, we use the scalar products of the vector $\mathbf{P} = (X, Y)$ of the right-hand sides of the desired system of equations (1.1). and the vectors $\mathbf{n}$ and $\boldsymbol{\tau}$,

$$F_1 = n_x X + n_y Y, \quad F_2 = \tau_x X + \tau_y Y \tag{3.4}$$

The functions F_1 and F_2 corresponding to the given set of singular trajectories (2.1) and the given topological structure of the partition of Ω into trajectories (Fig. 1) are expressed as products

$$F_1 = \xi_1 \prod_i \omega_i^{\alpha_i}, \quad F_2 = \xi_2 \prod_j \bar{\omega}_j^{\bar{\alpha}_j} \tag{3.5}$$

where α_i and $\bar{\alpha}_j$ are arbitrary natural numbers, and $\xi_1 = \xi_1(x, y)$, $\xi_2 = \xi_2(x, y)$ and $\bar{\omega}_j = \bar{\omega}_j(x, y)$ are arbitrary functions of the coordinates x and y that satisfy the condition

$$F_1^2 + F_2^2 \neq 0 \quad \text{in the domain} \quad \Omega \backslash (A_1 \cup A_2 \cup A_3) \tag{3.6}$$

The functions ξ_1, ξ_2 and $\bar{\omega}_j$ are selected in two steps. At the first step one establishes the structure of these functions in the neighbourhood of each of the equilibrium state A_k. That is done using theorems from [11], which contain algorithms for constructing these factors and sufficient conditions for the desired system of equations to have the given local topological structure in the equilibrium state specified. The proofs of these theorems are based on Frommer's method for investigating the singular points of differential equations [15]. The functions thus obtained, ξ_1, ξ_2 and $\bar{\omega}_j$, are used at the second step to form the final expressions of the functions F_1 and F_2, corresponding to which we have a system of equations

$$\dot{x} = (F_1 n_x + F_2 n_y)/(n_x^2 + n_y^2), \quad \dot{y} = (F_1 n_y - F_2 n_x)/(n_x^2 + n_y^2) \tag{3.7}$$

for which the partition of the domain Ω into trajectories has the given topological structure.

Step 1. Constructing of the F-functions of the equilibrium states.

The neighbourhood of the point $A_1(0, 0)$. The curve $\omega_1 = 0$ is a separatrix of hyperbolic type of the equilibrium state A_1, while the normal domains of the exceptional direction $y = 0$ of A_1 are normal domains of mixed type (Fig. 1). Based on Theorems 5 and 11 of [11], we define

$$F_1 = \xi_{11}\omega_1\omega_2^2\omega_{\upsilon 11}, \quad F_2 = \xi_{21}\bar{\omega}_1\omega_{\upsilon 21}(x^4+y^2), \quad \xi_{11}, \xi_{21} > 0 \tag{3.8}$$

where

$$\bar{\omega}_1 = x+y, \quad \omega_{\upsilon 11} = x+\zeta_1 y^{\eta_1}, \quad \omega_{\upsilon 21} = x+\zeta_2 y^{\eta_2}, \quad \eta_1 = \eta_2 = 3$$

and the quantities ζ_1 and ζ_2 are determined from the condition that the function

$$\xi_{1k}(\upsilon+\zeta_1)(b_1-c_1\upsilon^3)+\xi_{2k}(\upsilon+\zeta_2)(3b_1-2c_1\upsilon^3)$$

has no zeros. Throughout this section, a_k, b_k and c_k denote the coefficients of the principal terms in the expansions of the expression

$$\sum_i \lambda_i n_{ix} \prod_{j, j\neq i} \omega_j^{\alpha_{ij}}, \quad \sum_i \lambda_i n_{iy} \prod_{j, j\neq i} \omega_j^{\alpha_{ij}}, \quad \mu_k \prod_i \omega_i^{\beta_{ki}}$$

which occur in the expressions for the coordinates of the vector $\mathbf{n}$, in powers of $y-y_k$ in the neighbourhood of the equilibrium state A_k along the direction $x-x_k = 0$ (for more details see [11]). In particular, setting

$$\zeta_1 = -(b_1/c_1)^{1/3}, \quad \zeta_2 = -(3b_1/(2c_1))^{1/3}$$

we obtain the functions

$$F_1 = \xi_{11}\omega_1\omega_2^2[x-(b_1/c_1)^{1/3}y^3], \quad \xi_{11} > 0 \tag{3.9}$$

$$F_2 = \xi_{21}[x-y^3(3b_1/(2c_1))^{1/3}](x^4+y^2)\bar{\omega}_1, \quad \xi_{21} > 0 \tag{3.10}$$

corresponding to which is the system of equations (1.1) with the given local topological structure at A_1 (Fig. 1).

The neighbourhood of the point $A_2(0, 1)$. The curves $\omega_1 = 0$ and $\omega_2 = 0$ are separatrices of hyperbolic type of the equilibrium state A_2 (Fig. 1). In the neighbourhood of that point, relying on Theorems 1 and 11 of [11], we define

$$F_1 = \xi_{12}\omega_1\omega_3[x-(b_2/c_2)^{1/3}(y-1)^3], \quad \xi_{12} < 0 \tag{3.11}$$

$$F_2 = -\xi_{22}[x-(3b_2/(2c_2))^{1/3}(y-1)^3]\bar{\omega}_{21}\bar{\omega}_{22}, \quad \xi_{22} > 0 \tag{3.12}$$

where

$$\bar{\omega}_{21} = x+y-1, \quad \bar{\omega}_{22} = x-y+1$$

Corresponding to these functions F_1 and F_2 there is the system of equations (1.1) with the required local topological structure A_2 (Fig. 1).

The neighbourhood of the point $A_3(1, 1)$. The curve $\omega_3 = 0$ is a separatrix of hyperbolic type, while the curve $\omega_2 = 0$ corresponds to a normal domain of mixed type of the equilibrium state A_3 (Fig. 1). On the basis of Theorems 3, 5 and 11 of [11], we define in the neighbourhood of A_3

$$F_1 = \xi_{13}\omega_2^2\omega_3\omega_{\upsilon 31}, \quad \xi_{13} < 0 \tag{3.13}$$

$$F_2 = \xi_{23}[(x-1)^2+(y-1)^4]\{[y-1-3(x-1)]^2+(x-1)^4\}, \quad \xi_{23} > 0 \tag{3.14}$$

where

$$\omega_{v13} = x - 1 + \zeta_1 (y-1)^{\eta_1}, \quad \eta_1 = 3, \quad \zeta_1 = -a_3/c_3$$

Corresponding to these functions F_1 and F_2 is a system of equations (3.7) whose equilibrium point A_3 has the given local topological structure (Fig. 1).

The construction of the vector $\mathbf{n}$ is completed by choosing the values of the arbitrary factors λ_i, μ_k and α_{ij}, taking the principal terms in the expansions of the functions F_1 and F_2 (3.9)–(3.14) at the points A_i into account. In particular, we set

$$\begin{aligned}\mathbf{n} &= \lambda_{10}\mathbf{n}_1 L_{8,8,0}^{0,4,4} + \lambda_{20}\mathbf{n}_2 L_{4,0,4}^{4,0,4} + \lambda_{30}\mathbf{n}_3 L_{0,4,4}^{4,4,0} + \\ &+ \mu_{10}\mathbf{m}_1 L_{0,2,4}^{2,2,4} + \mu_{20}\mathbf{m}_2 L_{2,0,4}^{2,4,2} + \mu_{30}\mathbf{m}_3 L_{2,4,0}^{4,2,2}\end{aligned} \tag{3.15}$$

where

$$L_{p_1,p_2,p_3}^{s_1,s_2,s_3} = \omega_1^{s_1}\omega_2^{s_2}\omega_3^{s_3} r_1^{p_1} r_2^{p_2} r_3^{p_3}, \quad r_i^2 = (x-x_i)^2 + (y-y_i)^2$$

and λ_{i0} and μ_{i0} are arbitrary constants.

Note that in the neighbourhood of A_1 along the direction $x = 0$

$$\lambda_{10} n_{1x} L_{8,8,0}^{0,4,4} = \lambda_{10} y^{12} + o(y^{12}), \quad \mu_{10} L_{0,2,4}^{0,2,4} = 4\mu_{10} x^2 y^2 + o(x^2 y^2)$$

Hence it follows that

$$b_1 = \lambda_{10}, \quad c_1 = 4\mu_{10}$$

In the neighbourhood of A_2 along the direction $x = 0$

$$\lambda_{10} n_{1x} L_{8,4,0}^{0,4,4} = \lambda_{10}(y-1)^{12} + o((y-1)^{12}), \quad \mu_{20} L_{2,0,4}^{0,4,2} = \mu_{20}(y-1)^2 + o((y-1)^2)$$

Thus

$$b_2 = \lambda_{10}, \quad c_2 = \mu_{20}$$

In the neighbourhood of A_3 along the direction $x - 1 = 0$, we have

$$n_x = 2\mu_{30}(y-1)^4 + 4(4\lambda_{10} - 3\lambda_{20})(y-1)^8 + o((y-1)^8)$$

$$n_y = 2\mu_{30}(y-1)^4 + (4\lambda_{20} + \lambda_{30})(y-1)^8 + o((y-1)^8)$$

in the expression (3.15), and consequently

$$a_3 = 4(4\lambda_{10} - 3\lambda_{20}), \quad b_3 = 4\lambda_{20} + \lambda_{30}, \quad c_3 = 2\mu_{30}$$

To fix our ideas, we set

$$\lambda_{10} = \lambda_{20} = \lambda_{30} = 2, \quad \mu_{10} = 1/4, \quad \mu_{20} = 1, \quad \mu_{30} = 1/2 \tag{3.16}$$

and we obtain

$$b_1 = b_2 = 2, \quad c_1 = c_2 = c_3 = 1, \quad a_3 = 8, \quad b_3 = 10$$

After these quantities have been substituted into relations (3.9)–(3.14), we find the functions F_1 and F_2 corresponding to the given local topological structures of the equilibrium states A_i

$$A_1:\begin{cases} F_1 = \xi_{11}\omega_1\omega_2^2(x-2^{1/3}y^3), & \xi_{11}>0 \\ F_2 = \xi_{21}\bar{\omega}_1(x-3^{1/3}y^3)(x^4+y^2), & \xi_{21}>0 \end{cases}$$

$$A_2:\begin{cases} F_1 = \xi_{12}\omega_1\omega_3[x-2^{1/3}(y-1)^3], & \xi_{12}<0 \\ F_2 = \xi_{22}\bar{\omega}_{21}\bar{\omega}_{22}[x-3^{1/3}(y-1)^3], & \xi_{22}>0 \end{cases} \tag{3.17}$$

$$A_3:\begin{cases} F_1 = \xi_{13}\omega_2^2\omega_3[(x-1)-8(y-1)^3], & \xi_{13}<0 \\ F_2 = \xi_{23}[(x-1)^2+(y-1)^4]\{[(y-1)-3(x-1)]^2+(x-1)^4\}, & \xi_{23}>0 \end{cases}$$

where

$$\bar{\omega}_1 = x+y, \quad \bar{\omega}_{21} = x+y-1, \quad \bar{\omega}_{22} = x-y+1$$

Step 2. Construction of F- functions in the domain Ω as a whole.

Using expression (3.17), we form a graphic scheme (Fig. 2) illustrating the relative positions in Ω of the curves at whose points the required functions F_1 and F_2 vanish. One of the main conditions imposed on these functions is that they satisfy inequality (3.6), which is equivalent to the condition that the curves $F_1 = 0$ and $F_2 = 0$ have no points of intersection in Ω other than A_i. This condition holds, in particular, for the functions

$$\begin{aligned} F_1 &= \xi_1\omega_1\omega_2^2\omega_3\omega_{\upsilon 31}^*\omega_1^*\omega_2^*, \quad \xi_1<0 \\ F_2 &= \xi_2\bar{\omega}_1\bar{\omega}_{22}\omega_{\upsilon 22}\bar{\omega}_1^* r_1^4[(x-1)^2+(y-1)^4]\{[y-1-3(x-1)]^2+(x-1)^4\}, \quad \xi_2>0 \end{aligned} \tag{3.18}$$

The factors ω_1^*, $\omega_{\upsilon 31}^*$, $\bar{\omega}_1^*$ are such that

(1) the Taylor expansions of ω_1^* at the points A_1 and A_2 are identical, to within terms of order up to and including three, with the analogous expansions of the factors $\omega_{\upsilon 11}$ and $\omega_{\upsilon 12} \equiv x - 2^{1/3}(y-1)^3$, respectively;

(2) the curve $\omega_{\upsilon 31}^* = 0$ has no points in common with the domain Ω other than A_3, where its Taylor expansion has the form $\omega_{\upsilon 31}^* = (x-1) - 8(y-1)^3 + \ldots$;

(3) the curve $\bar{\omega}_1^* = 0$ (Fig. 2) approximates the curves $\omega_{\upsilon 21}$ and $\bar{\omega}_{21} = 0$ at the equilibrium states A_1 and A_2, respectively.

In particular, we set

$$\begin{aligned} \omega_1^* &= \left\{\left(y^4+\frac{1}{2}\right)(y-1)^4+\left[(y-1)^4+\frac{1}{2}\right]y^4\right\}x-2^{1/3}y^3\left(y-\frac{1}{2}\right)(y-1)^3 \\ \omega_{\upsilon 31}^* &= [1+28(y-1)^2](x-1)-8(y-1)^3, \quad \bar{\omega}_1^* = x+3^{1/3}y^3(y-1) \\ \omega_2^* &= x-\frac{1}{2}, \quad \omega_{\upsilon 22} = x-3^{1/3}(y-1)^3 \end{aligned} \tag{3.19}$$

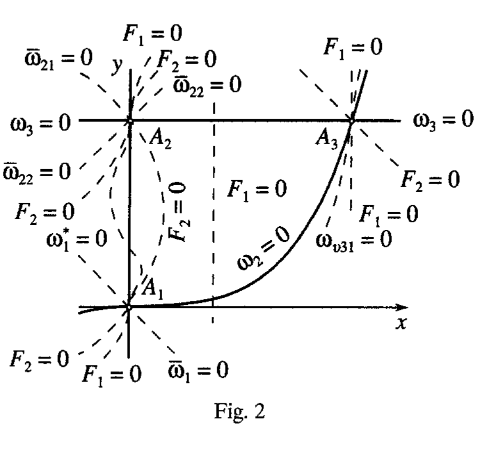

Fig. 2

The fact that the relative positions of these curves and the curves $\omega_i = 0$ correspond to the scheme of Fig. 2 has been verified by constructing their graphs using the MAPLE V software package.

Assuming that the coefficients in (3.15) satisfy equalities (3.16), we replace the coordinates of the vectors $\mathbf{n}_i$ and $\mathbf{m}_j$ by their explicit expressions (3.1) and (3.2). As a result, we obtain the final expressions for the coordinates of the vector $\mathbf{n}$

$$\begin{aligned} n_x &= 2L_{8,8,0}^{0,4,4} - 6x^2L_{4,0,4}^{4,0,4} - \frac{1}{4}xyL_{0,2,4}^{2,2,4} - x(y-1)L_{2,0,4}^{2,4,2} - \frac{1}{2}(x-1)(y-1)L_{2,0,4}^{4,2,2} \\ n_y &= 2L_{4,0,4}^{4,0,4} + 2L_{0,4,4}^{4,4,0} + \frac{1}{4}x^2L_{0,2,4}^{2,2,4} + x^2L_{2,0,4}^{2,4,2} + \frac{1}{2}(x-1)^2L_{2,4,0}^{4,2,2} \end{aligned} \tag{3.20}$$

Substituting these expressions n_x and n_y into system (3.7), together with F_1 and F_2 from formulae (3.17) and $\xi_1 = -(n_x^2 + n_y^2)$, $\xi_2 = n_x^2 + n_y^2$, we find the right-hand sides of the system of equations (1.1) for which the partition of Ω into trajectories (Fig. 2) has the given topological structure and which constitute the kinematic equations of motion of the point M.

The correctness of the solution has been verified by graphical constructions (using the MAPLE V software package Release 5) of the vector fields of directions $\boldsymbol{\tau}$, $\mathbf{n}$ and $\mathbf{P}(X, Y)$ corresponding to the expressions n_x, n_y, X and Y just found.

4. CONSTRUCTION OF THE CONTROLS

The system of equations (1.1), whose right-hand sides we have just constructed, defines a vector field of velocities of motions of the point M which guarantees that the objective of the control in the problem formulated in Section 2 will be accomplished. The components of the accelerations of these motions

$$\ddot{x} = \partial_x X\dot{x} + \partial_y X\dot{y} \quad (x \leftrightarrow y, X \leftrightarrow Y) \tag{4.1}$$

are caused by the action on M of a force $\mathbf{F}(F_x, F_y)$:

$$F_x = m(\partial_x XX + \partial_y YY) \quad (x \leftrightarrow y, X \leftrightarrow Y)$$

For the practical implementation of the programmed motion of M in the domain Ω, it is more convenient to apply a force with coordinates

$$F_x = m(\partial_x X\dot{x} + \partial_y X\dot{y}) + \Phi_x \quad (x \leftrightarrow y, X \leftrightarrow Y) \tag{4.2}$$

at the point, where $\boldsymbol{\Phi}(\Phi_x, \Phi_y)$ is a correcting force that reduces the measure of the deviation $\omega = (\dot{x} - X)^2 + (\dot{y} - Y)^2$ of the velocity of motion of M from the value determined by the equations of system (1.1). In addition, the force $\boldsymbol{\Phi}$ should be chosen so that the system of equations

$$\ddot{x} = \partial_x X\dot{x} + \partial_y X\dot{y} + \Phi_x/m \quad (x \leftrightarrow y, X \leftrightarrow Y) \tag{4.3}$$

has equilibrium states in the domain $\Omega \times R^2$ of the phase space $(x, y, \dot{x}, \dot{y})$ which are located only in the Oxy plane and which are precisely the equilibrium states of the system of equations (1.1) in the domain Ω.

It follows from the expression

$$(d\omega/dt)|_{(4.3)} = \frac{2}{m}[(\dot{x} - X)\Phi_x + (\dot{y} - Y)\Phi_y]$$

for the total derivative of ω with respect to time, evaluated along trajectories of the equations of system (4.3), that ω tends asymptotically to zero in time if

$$(\dot{x} - X)\Phi_x + (\dot{y} - Y)\Phi_y = -\lambda\omega^\sigma \tag{4.4}$$

where σ is a natural number and λ is an arbitrary non-negative function which can only vanish at the points where $\omega = 0$. Such a choice of Φ_x and Φ_y is possible. In particular, we set

$$\Phi_x = -\frac{m\lambda_0}{2}(\dot{x} - X), \quad \Phi_y = -\frac{m\lambda_0}{2}(\dot{y} - Y), \quad \lambda_0 = \text{const} > 0 \tag{4.5}$$

corresponding to the values $\sigma = 1$ and $\lambda = \lambda_0$ in (4.4). It follows from the representation of system (4.3) in the form

$$\dot{x} = p, \quad \dot{p} = \partial_x X\dot{x} + \partial_y X\dot{y} - \frac{\lambda_0}{2}(\dot{x} - X) \quad (x \leftrightarrow y, X \leftrightarrow Y, p \leftrightarrow q) \tag{4.6}$$

that, if $\dot{x} = \dot{y} = \dot{p} = \dot{q} = 0$, then necessarily $X(x, y) = Y(x, y) = 0$. This means that there are equilibrium states in the domain $\Omega \times R^2$ of the phase space (x, y, p, q) only in the Oxy plane, and that these are all equilibrium states of the corresponding system of equations (1.1) in the domain Ω. The converse is also true: corresponding to each equilibrium state (x_i, y_i) of system (1.1) in the domain Ω there is the equilibrium state $(x_i, y_i, 0, 0)$ of system (4.6) in the domain $\Omega \times R^2$ of the phase space.

We will now consider the influence of the terms Φ_x and Φ_y defined by (4.5) on the structure of the equilibrium states $A_i(x_i, y_i, 0, 0)$ of system (4.3) in two cases.

1. The right-hand sides of the equations of system (1.1) have the following form in the neighbourhood of the equilibrium state $A_i(x_i, y_i)$

$$X = a(x - x_i) + b(y - y_i) + o(r_i), \quad Y = c(x - x_i) + d(y - y_i) + o(r_i)$$

where a, b, c and d are constants such that $ad - bc \neq 0$. In that case, comparison of the characteristic equations

$$\chi(k) \equiv (k - a)(k - d) - bc = 0, \quad k^2\chi(0) = 0 \quad \left(k - \frac{\lambda_0}{2}\right)^2 \chi(k) = 0$$

of system (1.1), (4.1) and (4.6), respectively, implies that the addition of the terms Φ_x and Φ_y to the right-hand sides of system (4.1), first does not affect the nature of the behaviour of the projections of its representative points onto the plane $p = 0$, $q = 0$ in the neighbourhood of the point $(x_i, y_i, 0, 0)$ of the phase space; and second, it guarantees the existence of a pair of negative roots of the characteristic equation, corresponding to the coordinates p and q.

2. The expansions of the right-hand sides X and Y of system (1.1) in powers of $x - x_i$ and $y - y_i$ do not contain linear terms. In that case, system (4.6) has the following form in the neighbourhood of the equilibrium state $A_i(x_i, y_i\ 0, 0)$

$$\dot{x} = p, \quad \dot{p} = -\frac{\lambda_0}{2}(p - X) \quad (x \leftrightarrow y, X \leftrightarrow Y, p \leftrightarrow q) \tag{4.7}$$

and its characteristic equation is $k^2(k - \lambda_0/2)^2 = 0$. This equation, and the fact that the total derivative $d\omega/dt$, evaluated along trajectories of system (4.7), equals $-\lambda_0\omega$, imply that as $t \to +\infty$ (or $t \to -\infty$) the following estimates hold

$$\omega \sim \exp(-\lambda_0 t), \quad x \sim t^{-1/n}, \quad y \sim t^{-1/n}$$

where $n + 1$ is the order of the expansions of the functions X and Y in powers of $x - x_i$ and $y - y_i$. Since the rate at which the function ω tends to zero in time significantly exceeds the rate at which the coordinates x and y tend to x_i and y_i, we may conclude that the projections of the representative points of the system onto the plane $p = 0$, $q = 0$ of the phase space (x, y, p, q) behave in a manner similar to that of the solutions of system (1.1) in the neighbourhood of the corresponding point A_i.

After substituting expressions (4.5) into relation (4.2), we find the components of the force

$$F_x = m\left[\partial_x X\dot{x} + \partial_y X\dot{y} - \frac{\lambda_0}{2}(\dot{x} - X)\right] \quad (x \leftrightarrow y, X \leftrightarrow Y)$$

guaranteeing that the objective of the control of the motion of the point mass M will be achieved.

5. APPENDIX

The motions of the point governed by Eqs (1.1) with the right-hand sides constructed in Section 3, as it moves from the initial position $A_1(0, 0)$ (Fig. 1) to its final position $A_3(1, 1)$, are infinitely long. In that connection, consider the system of equations

$$\dot{x} = f_1^{2/3} X(x, y), \quad \dot{y} = f_3^{2/3} Y(x, y) \tag{4.8}$$

where

$$f_i \equiv a_i x + b_i y + c_i \quad (a_i, b_i, c_i - \text{const}; \ i = 1, 3)$$

and the straight line $f_i = 0$ intersects the segment A_iA_2 of the straight line $\omega_i = 0$ and the arc A_1A_3 of the curve $\omega_3 = 0$ at points of some fairly small ε-neighbourhood of A_1 = when $i = 1$ and of A_3 when $i = 3$. In the interior of the domain bounded by the curves $\omega_i = 0$, the system of equations (4.8) and the corresponding system of equations (1.1) define the same trajectories, but the point moves along them at velocities of different magnitudes. In particular, at points of the straight lines $f_1 = 0$ and $f_3 = 0$, which lie in the interior of the domain bounded by the curves $\omega_i = 0$, the coordinates of the velocities $\dot{x}$ and $\dot{y}$ are zero, but these points themselves are not equilibrium points of system (4.8), since the solution of the equation $\dot{x} = x^{2/3}$ is $x^{1/3} = (t + C)/3$, where C is a constant of integration. Consequently, the time taken by the representative point in the interior of that domain to move from a point of the straight line $f_1 = 0$ to the straight line $f_3 = 0$ is finite, while the solution presented above of the problem formulated in Section 2 may be used to solve the problem of finding a control force that will steer the point mass in a finite time, without leaving Ω, from a given initial position M_0 on the straight line $f_1 = 0$ in a fairly small neighbourhood of A_1 to a final position M_k, which is a point of the straight line $f_3 = 0$ in a fairly small neighbourhood of A_3; moreover, that is done in such a way that the initial and final velocities of the controlled motion of the point are zero.

6. CONCLUSION

There are further advantages in using the method proposed in this paper to solve the problem formulated in Section 2:

(1) The right-hand sides of Eqs (1.1) may include arbitrary functions (known as Yerugin functions) which do not affect the topological structure of the required systems of equations of type (1.1) and may be used to impart additional properties to the motions of the controlled objects they describe.

(2) Equations (1.1), which describe the objective of the motion, may be integrated numerically without having to use inequalities to formulate the conditions for the controlled point to be situated in the domain of admissible positions and to verify those conditions at each step of the integration, since Eqs (1.1) define a vector field of velocities that guarantees that the objective is achieved from any admissible position of the point; in addition, one obtains feedback information from the coordinates:

Remarks. 1. The problem considered (Section 2) may be complicated by assuming that the domain Ω contains obstructions, bypassing which the point M must move from an initial position to a final position while satisfying prescribed conditions (Fig. 3); the theoretical assumptions necessary to that end, as well as algorithms, were described in [11].

2. The solution of the problem of the directed motion of a point may be used in combination with the decomposition method [16] for the analytical construction of equations for the programmed motions of the modules of manipulating systems and to synthesize controls which guarantee the implementation of those motions.

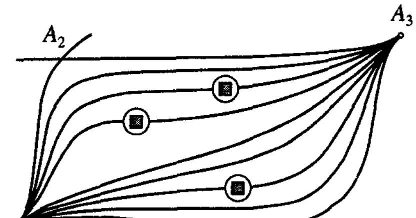

Fig. 3

REFERENCES

1. KORENEV, G. V., *Outline of the Mechanics of Directed Motion*. Nauka, Moscow, 1980.
2. GALIULLIN, A. S., *Inverse Problems of Dynamics*. Nauka, Moscow, 1981.
3. GALIULLIN, A. S., *Methods for Solving Inverse Problems of Dynamics*. Nauka, Moscow, 1986.
4. AL'MUKHAMEDOV, M. I., The inverse problem of the qualitative theory of differential equations. *Izv. Vuzov. Matematika*, 1963, 4, 3–6.
5. AL'MUKHAMEDOV, M. I., The construction of a differential equation whose limit cycles are given curves. *Izv. Vuzov, Matematika*, 1965, 1, 12–16.
6. MUKHARLYAMOV, R. G., Inverse problems of the qualitative theory of differential equations. *Differents Uravn*., 1967, **3**, 10, 1673–1681.
7. GALIULLIN, A. S., The construction of equations of motion. *Differents. Uravn.*, 1977, **13**, 2, 192–237.
8. GALIULLIN, A. S., MUKHARLYAMOV, R. G. and FURASOV, V. D., *The Construction of the Equations of Programmed Motion of Controllable Systems*. Izd. UDN, Moscow, 1969.
9. GALIULLIN, A. S., (Ed.), *The Programmed Motion of Mechanical Systems*. Textbook. Izd. UDN, Moscow, 1971.
10. GALIULLIN, A. S., (Ed.), *Differential Equations and Inverse Problems of Dynamics*. Izd. UDN, Moscow, 1983.
11. VOLKOV, S. V., *The Construction of Differential Operators of Dynamical Systems*. Izd. RUDN, Moscow, 1999.
12. NEMYTSKII, A. A. and STEPANO, V. V., *The Qualitative Theory of Differential Equations*. Gostekhizdat, Moscow, 1949.
13. ANDRONOV, A. A., LEONTOVICH, Ye. A., GORDON, I. I. and MAIER, A. G., *The Qualitative Theory of Second-order Dynamical Systems*. Nauka, Moscow, 1966.
14. YERUGIN, N. P., The construction of the set of all the systems of differential equations that have a given integral curve. *Prikl. Mat. Mekh.*, 1952, **16**, 6, 659–670.
15. FROMMER, M., Die Integralkurven einer gewöhnlichen Differentialgleichung erster Ordung in der Umgebung rationaler Unbestimmtheitsstellen. *Math. Ann.*, 1928, **99**, 222–272.
16. PYATNITSKII, Ye. S., The principle of decomposition in the control of mechanical and electromechanical systems. In *The Synthesis of the Control Systems of Robot Manipulators Based on the Decomposition Principle*. Inst. Problem Upravleniya, Moscow, 1987, 4–15.

Translated by D.L.